

# Non-commutative heat kernel

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## Abstract

We consider a natural generalisation of the Laplace type operators for the case of non-commutative (Groenewold-Moyal star) product. We demonstrate existence of a power law asymptotic expansion for the heat trace of such operators on  $T^n$ . First four coefficients of this expansion are calculated explicitly. We also find an analog of the UV/IR mixing phenomenon when analysing the localised heat kernel.

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## 1 Introduction

Quantum field theory on non-commutative spaces (see reviews [9, 16]) is a very fast developing topic. Although the heat kernel expansion is an essential ingredient of some earlier approaches to the non-commutative field theory [5] properties of the heat expansion of operators on non-commutative spaces remain a relatively neglected subject. We can mention a recent work [18] which studied  $q$ -deformed zeta functions (without relation to a particular operator, however).

At the same time, the heat trace expansion (which is also called the heat kernel expansion in the physical literature) [11, 13, 19] is a very powerful instrument of ordinary (commutative) quantum field theory. In particular, coefficients of this expansion define the one-loop counterterms, quantum anomalies, and various expansions of the effective action (e.g., the large mass expansion).

The aim of this paper is to study the asymptotics of the heat trace for a natural non-commutative generalisation of the Laplace type operator. Roughly speaking,

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this generalisation is achieved by replacing ordinary product by the Groenewold-Moyal star (cf. eqs. (2) and (3) below). Because of the presence of the Groenewold-Moyal star the operator does not fall into the category considered by Seeley [17] (see [12] for a recent review). Therefore, very little is known about behaviour of main spectral functions for such operators. In the next section we show that somewhat surprisingly the heat trace on a torus admits a power law asymptotic expansion for small proper time  $t$ . The coefficients of this expansion can be easily calculated. We present explicit expression for first four coefficients. Our main message is that the heat trace coefficients for the non-commutative case are fully defined by the heat trace expansion for ordinary (“commutative”) but non-abelian operators. In section 3 we analyse the localised heat trace and find an analog of the so-called UV/IR mixing phenomenon.

## 2 Heat trace asymptotics

Let us consider an  $n$ -dimensional torus  $T^n$  with the coordinates  $0 \leq x^j < 2\pi r_j$ ,  $j = 1, \dots, n$ . A non-commutative version of  $T^n$  (also denoted as  $T_\theta^n$ ) is constructed in the following way [7]. One considers a non-commutative associative  $*$ -algebra with unit generated by  $n$  elements  $U_j$  which obey the relation

$$U_j U_l = e^{-i\theta^{jl}/(r_j r_l)} U_l U_j \quad (1)$$

with some antisymmetric matrix  $\theta^{jl}$ . The “completion” of this algebra consists of formal power series with sufficiently fast decreasing coefficients. There is a one-to-one correspondence between elements of these algebra and complex valued functions on  $T^n$  so that  $U_j$  is identified with  $e^{ix^j/r_j}$  and the relation (1) is induced by the Groenewold-Moyal star product

$$f \star g = f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x). \quad (2)$$

In this form the star product has to be applied to plane waves and then extended to all (square integrable) functions by means of the Fourier series. This is exactly the way the star product will be treated in this paper. We note that (2) arises in the context of deformations of flat Poisson manifolds [2] (see also recent reviews [8, 10]).

In this paper we consider a natural generalisation of the Laplace type operators for the non-commutative case:

$$D\phi = - \left( \delta^{\mu\nu} \partial_\mu \partial_\nu + a^\mu \partial_\mu + b \right) \star \phi. \quad (3)$$

We shall call the operator (3) the star-Laplacian. We suppose that  $\phi$  is multi-component, and  $a^\mu$  and  $b$  are some matrix valued functions. In principle, this construction can be translated into the vector bundle language. The coefficient in front of the second

derivative term defines a Riemannian metric on  $T^n$  which is the unit one in the present case. Consequently, there is no distinction between upper and lower vector indices. The operator (3) can be represented in the canonical form:

$$D\phi = -(\delta^{\mu\nu}\nabla_\mu \star \nabla_\nu + E) \star \phi, \quad (4)$$

where

$$\nabla_\mu = \partial_\mu + \omega_\mu, \quad \omega_\mu = \frac{1}{2}a_\mu, \quad E = b - \partial^\mu \omega_\mu - \omega^\mu \star \omega_\mu. \quad (5)$$

It is convenient to introduce the field strength of  $\omega$ :

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + \omega_\mu \star \omega_\nu - \omega_\nu \star \omega_\mu. \quad (6)$$

The inner product in the space of fields is not sensitive to the non-commutativity parameter:

$$\langle \psi, \phi \rangle = \int d^n x \psi^\dagger \phi = \int d^n x \psi^\dagger \star \phi \quad (7)$$

for smooth  $\psi$  and  $\phi$  (meaning that this star product is closed). One can easily check that the operator  $D$  is hermitian if  $E(x)$  is a hermitian matrix, and  $\omega_\mu(x)$  is an anti-hermitian matrix at each point.

Let us now turn to the heat trace which is defined as a functional trace in the space of square integrable functions on  $M$ :

$$K(t, D) = \text{Tr}_{L^2}(\exp(-tD)). \quad (8)$$

If  $D = D_0$  is a partial differential operator of Laplace type (which is achieved in the limit  $\theta \rightarrow 0$ ) the heat kernel is well defined for positive  $t$  and there is an asymptotic series as  $t \rightarrow +0$ :

$$K(t, D_0) \cong \sum_{k \geq 0} t^{(k-n)/2} a_k(D_0). \quad (9)$$

If  $M$  has no boundaries, odd-numbered coefficients vanish,  $a_{2j+1} = 0$ . Moreover, the coefficients  $a_k$  are locally computable, i.e. they can be presented as integrals over  $M$  of some local invariants constructed from  $\nabla_\mu$  and  $E$ . In Quantum Field Theory language this corresponds to locality of the counterterms. We stress, that all these properties hold only in the limit  $\theta \rightarrow 0$ .

For non-zero values of  $\theta$  the operator  $D$  is a second order differential operator on the non-commutative torus. To analyse spectral functions of this operator we need a non-commutative version of the pseudo-differential calculus (which is available in some form [7]), but also a non-commutative version of the symbolic calculus (which, roughly speaking, tells how one multiplies symbols of the operators). This second tool is not available. Therefore, we adopt a different point of view widely accepted in the quantum field theory context by considering the star multiplication by a function

as being a pseudodifferential operator on a commutative torus. The formula (2) is valid as it stays for the Fourier harmonics only, but this is exactly what we need in the pseudodifferential context. Then  $D$  itself is a pseudodifferential operator ( $\psi$ do) rather than a differential one. The study of spectral geometry of  $\psi$ do's was initiated by Seeley [17] (see [12] for an overview). In particular, it was shown that  $\ln t$  terms can appear in the heat kernel expansion and the heat trace coefficients become, in general, non-local. However, even these results are not applicable to our case since the symbol of  $D$  does not belong to the so-called standard symbol space<sup>1</sup>.

To proceed further we need the following definition. We call a functional of  $\nabla_\mu$  and  $E$  a star-local polynomial functional if it is an integral over  $M$  of a finite sum of monomials each consisting of a star product of a finite number of  $\nabla_\mu$  and  $E$  taken in an arbitrary order. For example, integrals of  $E$  and of  $\Omega_{\mu\nu} \star \Omega^{\mu\nu}$  are star local polynomial functionals, while that of  $E^3$  is not. In other words, these functionals are integrals of free polynomials of  $E$ , the curvature coefficients and their covariant derivatives (evaluated in the star-algebra). This definition will be useful since we have two different multiplications in the game.

Before analysing the heat trace asymptotics one has to make sure that the heat trace exists for positive  $t$ . For a physicist, the exponential damping established below is probably enough to establish the existence. There is also a mathematical proof<sup>2</sup>.

Now we can formulate our main result. Let  $D$  be a star-Laplacian (4) on  $T^n$ . Then

1. There is a power-law asymptotic expansion (9) of the heat kernel for the operator  $D$ . The coefficients  $a_k(D)$  are star-local polynomial functionals<sup>3</sup> of  $\nabla$  and  $E$ .

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<sup>1</sup>Roughly speaking, the symbol  $A(x, \xi)$  is obtained from a  $\psi$ do  $D$  by replacing all partial derivatives by  $i\xi$ , like in doing the Fourier transform. For the symbols belonging to the standard symbol space  $\partial_\xi^\alpha A(x, \xi) = \mathcal{O}\left((1 + |\xi|^2)^{(m-|\alpha|)/2}\right)$ , where  $m$  is the order of  $D$  and  $\alpha$  is a multi-index. In other words, each derivative with respect to  $\xi$  improves behaviour of  $A$  at  $\xi \rightarrow \infty$ . This condition clearly excludes oscillatory behaviour of the symbol.

<sup>2</sup>Such a proof was suggested by the referee of this paper. It consists of the following main steps. It is easily seen that the operator  $A = D - D_0$  is of degree 1 in the Sobolev scale ( $AH^s \subset H^{s-1}$ ). This also ensures existence of the semigroup  $e^{-tD}$ . Then if  $(\partial_t - D)f = 0$ ,  $f \in H^{r,s}$  ( $H^{r,s}$  consists of functions with  $r$  square integrable derivatives w.r.t.  $t$  and  $s$  w.r.t.  $x$ ), we have  $Af, D_0f \in H^{r,s-1}$ , so  $f \in H^{r+1,s+1}$ . Thus for all  $t > 0$ ,  $e^{-tD}$  is a continuous map  $L^2 \rightarrow C^\infty$  so it is of trace class.

<sup>3</sup>This also implies that all dependence on  $\theta$  is hidden in the star product. Such property cannot be supported in the approaches which consider perturbative expansions in  $\theta$ .

2. In particular,

$$a_0 = (4\pi)^{-n/2} \text{tr}(I) \text{volume } T^n, \quad (10)$$

$$a_2 = (4\pi)^{-n/2} \int d^n x \text{tr}(E), \quad (11)$$

$$a_4 = (4\pi)^{-n/2} \frac{1}{12} \int d^n x \text{tr} (6E \star E + \Omega^{\mu\nu} \star \Omega_{\mu\nu}) \quad (12)$$

$$a_6 = (4\pi)^{-n/2} \frac{1}{360} \int d^n x \text{tr} (60E \star E \star E + 30E \star E_{;\mu\mu} + 30E \star \Omega_{\mu\nu} \star \Omega_{\mu\nu} - 4\Omega_{\mu\nu;\rho} \star \Omega_{\mu\nu;\rho} + 2\Omega_{\mu\nu;\nu} \star \Omega_{\mu\rho;\rho} - 12\Omega_{\mu\nu} \star \Omega_{\nu\rho} \star \Omega_{\rho\mu}), \quad (13)$$

where semicolon denotes covariant differentiation,  $E_{;\mu} := \partial_\mu E + \omega_\mu \star E - E \star \omega_\mu$ .  $\text{tr}$  is the matrix trace.

Proof consists in a rather straightforward evaluation of the asymptotic behaviour of (8) (cf sec. 4.1 of ref. [19]). To calculate the trace in (8) we need a basis in  $L^2$  on the torus. Let  $\{u^a\}$  be a basis in the “internal” space. Then the functions

$$\phi_k^a(x) = \frac{u^a e^{ikx}}{(2\pi)^{n/2} (r_1 r_2 \dots r_n)^{1/2}} \quad (14)$$

with  $\{\tilde{k}_\mu\} = \{k_\mu r_\mu\} \in \mathbb{Z}^n$  (no summation over  $\mu$ ) form an orthonormal system on the torus. One can represent the heat trace in the form:

$$K(t, D) = \int d^n x \sum_a \sum_k \phi_k^{a\dagger}(x) \exp(-tD) \phi_k^a(x). \quad (15)$$

Here the integral over  $x$  comes from the scalar product (7) needed to calculate diagonal matrix elements, the trace is then taken by summing over  $a$  and  $k$ . As a next step, we expand the exponential in (15) and push all derivatives to the right. Typical monomial obtained in this way reads:

$$a \star b \star \dots \star c \star (\partial)^{(\alpha)}, \quad (16)$$

where  $(\alpha)$  is a multi-index.  $a, b, c$  stay instead of  $E$  or  $\omega$ , or their derivatives. Let now the operator (16) act on  $e^{ikx}$ . One obtains:

$$e^{ikx} (a \star_k (b \star_k \dots \star_k (c \star_k 1)) \dots) (ik)^{(\alpha)}, \quad (17)$$

where

$$f \star_k g = f(x) \exp \left( \frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{(\partial_\nu + ik_\nu)} \right) g(x). \quad (18)$$

Since this binary operation is not associative we have had to put brackets in (17). Again, the formula (18) has to be understood through the Fourier series.

Let us expand  $a, b, c$  in Fourier series:  $a(x) = \sum_{q^{[a]}} a_{q^{[a]}} e^{iq^{[a]}x}$  etc. If the monomial (16) is sandwiched between  $\phi_k^\dagger(x)$  and  $\phi_k(x)$  the exponential  $e^{ikx}$  in (17) is cancelled, and the whole  $x$ -dependence resides in the phase factor:

$$\exp \left( i(q_\mu^{[a]} + q_\mu^{[b]} + \dots + q_\mu^{[c]})x^\mu \right). \quad (19)$$

If now we integrate over  $x$  as prescribed by (15), we obtain a delta-symbol:

$$\delta \left( q_\mu^{[a]} + q_\mu^{[b]} + \dots + q_\mu^{[c]} \right) \quad (20)$$

Next we note that the only effect of the modification (18) of the star product in (17) is the phase factor

$$\exp \left( -\frac{i}{2} \theta^{\mu\nu} (q_\mu^{[a]} + q_\mu^{[b]} + \dots + q_\mu^{[c]}) k_\nu \right) = 1, \quad (21)$$

where we have used (20). Therefore, we can as well delete the subscript  $k$  in the star products in (17). This technical observation turns out to be very important.

Next we collect again all monomials to an exponent. Up to this point we only used the definition of the exponential in terms of the power series forth and back and operated with finite sums or individual monomials. Let us perform the summation over  $a$  in (15). This yields:

$$K(t, D) = (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \dots r_n} \sum_{\tilde{k} \in \mathbb{Z}^n} \text{tr} \exp \left[ t \left( (\nabla_\mu + ik_\mu) \star (\nabla_\mu + ik_\mu) + E \right) \star \right]. \quad (22)$$

We stress again that the star product in (22) does not depend on  $k$ . Therefore, the asymptotic behaviour of (22) can be evaluated rather straightforwardly. One has to isolate  $e^{-tk^2}$  and expand the rest of the exponential in a power series. To justify this step one has to use the assumption above. For a physicists, the presence of the damping term  $-tk^2$  in the exponential is already enough to state that the sum over  $k$  is convergent for positive  $t$  and that the subsequent integral over  $x$  also exists at least for some “good”  $E$  and  $\omega$ . Then one sums over  $k$  by using the formulae:

$$\begin{aligned} \sum_{\tilde{k} \in \mathbb{Z}^n} e^{-tk^2} &\cong \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \dots r_n), \\ \sum_{\tilde{k} \in \mathbb{Z}^n} k_\mu k_\nu e^{-tk^2} &\cong \delta_{\mu\nu} \frac{1}{2t} \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \dots r_n), \\ \sum_{\tilde{k} \in \mathbb{Z}^n} k_\mu k_\nu k_\rho k_\sigma e^{-tk^2} &\cong (\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) \frac{1}{4t^2} \left( \frac{\pi}{t} \right)^{n/2} (r_1 r_2 \dots r_n), \end{aligned} \quad (23)$$

etc. Corrections to (23) are exponentially small if  $t \rightarrow +0$ . Clearly, in this way one obtains a power-law asymptotic expansion (9). If, as usual, one assigns dimension

one to  $\nabla$  and dimension two to  $E$ , simple power counting arguments show that each coefficient  $a_p$  is a star-local polynomial functional of  $E$  and  $\nabla$  with an integrand of the dimension  $p$ . This proves the first assertion formulated above in this section.

To illustrate how the procedure works, consider a simplified case  $E = 0$  and calculate the heat trace coefficients up to  $a_4$ .

$$K(t, D) = (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \dots r_n} \sum_{\tilde{k} \in \mathbb{Z}^n} \text{tr} e^{-tk^2} \exp \left[ t \left( \nabla_\mu \star \nabla_\mu + 2ik_\mu \nabla^\mu \right) \star \right]. \quad (24)$$

Now we expand the second exponential keeping only the terms of dimension four or lower:

$$\begin{aligned} K(t, D) \cong (2\pi)^{-n} \int \frac{d^n x}{r_1 r_2 \dots r_n} \sum_{\tilde{k} \in \mathbb{Z}^n} \text{tr} e^{-tk^2} & \left[ 1 + t \nabla^\mu \star \nabla_\mu \right. \\ & - 2t^2 (k\nabla) \star (k\nabla) + \frac{t^2}{2} \nabla^\mu \star \nabla_\mu \star \nabla^\nu \star \nabla_\nu \\ & - \frac{2t^3}{3} \left( (k\nabla) \star (k\nabla) \star \nabla^\mu \star \nabla_\mu + \nabla^\mu \star \nabla_\mu \star (k\nabla) \star (k\nabla) \right. \\ & \left. \left. + (k\nabla) \star \nabla^\mu \star \nabla_\mu \star (k\nabla) \right) + \frac{2t^4}{3} (k\nabla) \star (k\nabla) \star (k\nabla) \star (k\nabla) \right], \end{aligned} \quad (25)$$

where  $(k\nabla) := k^\mu \nabla_\mu$ . Next we use (23) to perform summation over  $k$ . All covariant derivatives combine into commutators yielding the final result:

$$K(t, D) \cong (4\pi t)^{-n/2} \int d^n x \text{tr} \left[ 1 + \frac{t^2}{12} \Omega_{\mu\nu} \star \Omega^{\mu\nu} + \mathcal{O}(t^3) \right]. \quad (26)$$

This result confirms (10) - (12) in the particular case considered. In principle one can go on and compute the rest of (10) - (13). However, this is not needed. The crucial fact is that the calculations go exactly the same way as in the commutative case (cf. sec. 4.1 of [19]). The reason is that even in a theory is commutative, both  $\omega$  and  $E$  are matrix-valued, and, therefore, commutativity is not being used in the course of the calculations. As a consequence, the heat trace coefficients (10) - (13) can be read off from the commutative but non-abelian results presented e.g. in [11, 13, 19].

### 3 Localised heat trace and UV/IR mixing

In ordinary commutative case the “global” heat trace (8) is sometimes replaced by a more general (localised) expression

$$K(f; t, D_0) = \text{Tr}_{L^2} (f \exp(-tD_0)), \quad (27)$$

where  $f$  is a function. Obviously,  $K(t, D_0) = K(1; t, D_0)$ . By varying (27) with respect to  $f(x)$  one obtains matrix elements of  $\exp(-tD_0)_{x,y}$  at coinciding arguments,  $x = y$ .

This modification proves convenient for technical reasons [11, 13]. More important is that (27) describes local quantum anomalies (cf. [19] and references therein).

A natural generalisation of (27) to the non-commutative case reads:

$$K_\star(f; t, D) = \text{Tr}_{L^2}(f \star \exp(-tD)) . \quad (28)$$

Let us consider the  $t \rightarrow +0$  asymptotic expansion of this expression. Clearly, all methods used in the previous section work also for this case. The only modification is to replace  $a$  by  $f$  in (16) and (17). Therefore, we conclude that there is an asymptotic expansion

$$K_\star(f; t, D) \cong \sum_{k=0,2,4,\dots} t^{(k-n)/2} a_k(f, D) , \quad (29)$$

where the coefficients  $a_k(f, D)$  are star-local polynomial functionals. Again, commutative non-abelian heat trace coefficients define uniquely the heat trace expansion for the non-commutative heat trace. One only has to remember to take matrix-valued smearing function  $f$  in the commutative case in order to preserve all relevant invariants<sup>4</sup>. In particular, by comparing with non-abelian commutative heat kernel coefficients given in [11, 13, 4] one easily derives first three coefficients in the expansion (29):

$$\begin{aligned} a_0(f, D) &= (4\pi)^{-n/2} \text{tr}(f) \text{volume } T^n , \\ a_2(f, D) &= (4\pi)^{-n/2} \int d^n x \text{tr}(f \star E) , \\ a_4(f, D) &= (4\pi)^{-n/2} \frac{1}{12} \int d^n x \text{tr} [f \star (2E_{;\mu\mu} + 6E \star E + \Omega^{\mu\nu} \star \Omega_{\mu\nu})] \end{aligned} \quad (30)$$

Of course, the same expressions may be obtained by more direct methods (cf. equations (24) - (26) above). One can make a simple but important observation that the “zero-momentum” limit  $f \rightarrow 1$  commutes with the asymptotic expansion in  $t$  so that  $a_k(D) = a_k(1, D)$ .

One can also consider a different generalisation of (27) to the non-commutative case:

$$K(f; t, D) = \text{Tr}_{L^2}(f[\exp(-tD)]) , \quad (31)$$

where there is no star between  $f$  and the exponent.

To evaluate an asymptotic expansion of (31) one can use again the same plane wave basis as in (15), but there is one important difference. Now we cannot replace the modified product  $\star_k$  by  $\star$  under the trace. The reason is that the momentum corresponding to  $f$  enters, of course, the momentum conservation delta function (cf. (20)) but does not appear in an analog of the phase factor (21). Therefore, one cannot guarantee existence of the power-law asymptotics and the star-locality. As we will see in a moment, all these properties are indeed violated.

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<sup>4</sup>One can calculate the heat trace expansion even if  $f$  is a differential operator [4].



To illustrate this point let us calculate (31) in the case of zero connection  $\omega_\mu = 0$  to the linear order in  $E$  neglecting also all derivatives of  $E$  which are not coupled directly to  $\theta$ . In this approximation

$$K(f; t, D) = \int d^n x \sum_a \sum_k \phi_k^{a\dagger}(x) f(x) \exp(-tk^2) (t E \star \phi_k^a(x)). \quad (32)$$

After expanding  $f$  and  $E$  in Fourier series, summing over  $k$ , and returning back to the coordinate representation one obtains:

$$K(f; t, D) = \frac{t}{(4\pi t)^{n/2}} \int d^n x \text{tr} \left[ E(x) \exp \left( \frac{\theta^{\mu\nu} \theta^{\mu\rho} \partial_\nu \partial_\rho}{16t} \right) f(x) \right] + \dots \quad (33)$$

where dots denote the higher order terms which we dropped in this calculation.

The differential operator in the exponential in (33) is non-positive. Therefore, for a non-constant  $f$  there is a very strong exponential damping at  $t \rightarrow 0$ . At first glance this exponential looks as a regulator of the heat kernel. However, the whole effect disappears for  $f = \text{const}$ . This is a manifestation of the so-called UV/IR mixing [14, 1, 6] which is a characteristic feature of non-commutative field theories. In the heat-kernel context this mixing is rather a consequence of the way we have made the localisation. The exponential factor in (33) is similar to the typical exponent  $\exp(-(x-y)^2/4t)$  which appears in matrix elements of the heat kernel of a “commutative” operator  $D_0$  between non-coinciding points  $x$  and  $y$ .

## 4 Conclusions

In this paper we considered a natural generalisation of the Laplace type operators for the non-commutative case (which we call the star-Laplacians). We have demonstrated that both global (8) and localised (29) heat traces for these operators on  $T^n$  admit a power-law asymptotic expansion for  $t \rightarrow +0$ . The coefficients of these expansions are star-local functionals of the  $\omega$  and  $E$ . They can be easily calculated. Expressions for fist several coefficients have been given explicitly. We have also considered a different way (31) to localise the heat trace and observed an analog of of the UV/IR mixing phenomenon. Most of the results of this paper may be reformulated for  $\mathbb{R}^n$  by imposing suitable fall-off conditions on the fields and on the function  $f$ .

We note that the star product is a natural object in the operator theory which describes composition of symbols of  $\psi$ do’s (see [12, 3]). It has been used in calculations of the effective action in *commutative* field theories [15]

Our results “confirm” in a way the spectral action principle [5]: the  $\Omega$ -term in  $a_4$  (cf. (12)) is just the action for non-commutative Yang-Mills theory.

Our results suggest that the background field formalism and spectral regularization methods (like, e.g., the zeta function regulation) are efficient tools to study divergences in non-commutative theories. Implications for renormalization of non-commutative field theories will be analysed in a separate publication.

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